Mathematics-IV (BT-405) — Exam Prep Pack

TU-829(A) · B.Tech IV Semester · Built from PYQ Pattern Analysis

For the night-before reader

Exam date: within the next few days

Contents

How to Use This Document

This pack is built from two BT-405 (TU-829(A)) papers — one from May 2024 and one from an earlier session. Both papers share the **exact same structure** and most question templates repeat. So this document is organised around what's *most likely to show up*, not around what's in the syllabus alphabetically.

Read in this order:

1. The **Pattern Analysis** section below — so you know what's worth your time.
2. The **Master Cheat Sheet** — memorise the formulas cold.
3. **Topic-by-Topic Teaching** — read top to bottom; topics are ordered by probability.
4. **Complete PYQ Solutions** — work through them with paper and pen. Don't just read.
5. **Back Matter (Quick Reference + Last-Minute Sheet)** — for the morning of.

A few stylistic notes:

- Sections marked with a quote block (>) contain text you should literally write in the answer book. They're tuned for the marks-per-question and the way examiners want the answer worded.
- The **mistakes box** at the bottom of each topic is where most marks bleed away in this paper. Read it twice.
- All diagrams were drawn for this pack — they are not from any textbook.

Part 1 — Pattern Analysis

The two papers side-by-side

Slot	Paper A (earlier)	Paper B (May 2024)	Verdict
Sec A · Q1	Form PDE from $z = f(x^2 + y^2)$	Form PDE from $z = f(x^2 + y^2)$	**Identical**
Sec A · Q2	Write 2-D heat equation	Write 1-D heat equation	Same family
Sec A · Q3	Write wave equation	Write wave equation	**Identical**
Sec A · Q4	Write Fourier Sine Transform	Write Lagrange's Linear Equation	Definitions slot
Sec A · Q5	P(odd number on a die)	P(2 heads in 3 tosses)	Simple probability
Sec B · Q6	Solve $Pq = 1$	Solve $p^2 + q^2 = 1$	Same Standard Form I
Sec B · Q7	Mean of grouped data	First moment of Binomial	Stats
Sec B · Q8	Urn — two balls same colour	Mean of grouped data	Stats / combinatorics
Sec C · Q9	$(D+1)(D+D'+1)z = \sin(x+2y)$	$(D+1)(D+D'-1)z = \sin(x+2y)$	**Same template**
Sec C · Q10	Charpit: $px + qy = pq$	Charpit: $(p^2+q^2)y = qz$	Same method
Sec C · Q11	Mode of age data	Mode of age data	**Same template**
Sec C · Q12	Bomb-strike Binomial	Correlation coefficient	Stats / probability
Sec C · Q13	Correlation + regression lines	Binomial with mean=4, var=2	Stats / probability

The "must prep" list — topics that appeared in BOTH papers

1. **Formation of PDE by elimination of arbitrary function**
2. **Heat and Wave equations** (definitions, derivations to lesser extent)
3. **Non-linear first-order PDE — Standard Form I**
4. **Linear PDE with constant coefficients** of the form `(D + α)(D + βD' + γ)z = sin/cos`
5. **Charpit's method**
6. **Mode and Mean of grouped data**
7. **Binomial distribution problems**
8. **Correlation coefficient** (and regression lines)
9. **Simple classical probability** (dice, coins, urns)

Predicted question paper for this year

Because both papers are templates of each other, the predictable Section A is almost a checklist. Below is the most-likely paper for the upcoming session.

Section A (5 × 2 = 10, all compulsory)

1. Form the PDE from `z = f(x² − y²)` or `z = x f(y/x)` *or* the unchanged `z = f(x² + y²)`.
2. Write the **one-dimensional heat equation** (it was 2-D in Paper A and 1-D in B, so 1-D is hot; Laplace equation is the dark horse).
3. Write the **one-dimensional wave equation** (or its solution `y = f₁(x+ct) + f₂(x−ct)`).
4. Write the **Fourier Cosine Transform** (sine appeared in A, Lagrange's in B; cosine is the next definition in the rotation).
5. Probability of getting **two heads in two tosses**, or two cards being the same colour from a pack — basic counting.

Section B (any 2 of 3 × 9 = 18)

6. Solve a Standard Form I or Form II PDE: e.g. `pq + p + q = 0` *or* a Clairaut form `z = px + qy + p² − q²`.
7. **Median of a grouped distribution** — mean came twice in a row, median is the natural rotation.
8. Binomial / Poisson short problem, OR an urn problem.

Section C (any 3 of 5 × 14 = 42)

9. `(D − 1)(D + D' + 2)z = sin(2x + y)` or similar — *same template* with new numbers.
10. **Charpit's method** on something like `2(z + xp + yq) = yp²` or `p²x + qy = z`.
11. **Mode** (or Median) of a grouped frequency table — different numbers, same structure.
12. **Correlation coefficient with regression lines** — practically guaranteed.
13. **Binomial with mean and variance** given, find P(X = k) probabilities.

Target score breakdown

Section	Strategy	Time	Aim
A (all 5)	All five — memorised cold	15 min	9/10
B (any 2 of 3)	PDE + safer stats Q	35 min	16/18
C (any 3 of 5)	Linear-PDE + Charpit + Correlation (or Mode)	100 min	35/42
Buffer	Recheck Charpit and the sin-PI	30 min	—
Total		**180 min**	**~60/70 (≈85%)**

Part 2 — Master Cheat Sheet (memorise cold)

PDE essentials

Symbol	Meaning
p	$\partial z/\partial x$
q	$\partial z/\partial y$
r	$\partial^2 z/\partial x^2$
s	$\partial^2 z/\partial x\partial y$
t	$\partial^2 z/\partial y^2$
D	$\partial/\partial x$
D'	$\partial/\partial y$

Equation	Form	Solution recipe
Lagrange's Linear PDE	$Pp + Qq = R$	Auxiliary: $dx/P = dy/Q = dz/R$
Standard Form I	$f(p, q) = 0$	$z = ax + by + c$ with $f(a, b) = 0$
Standard Form II (Clairaut)	$z = px + qy + f(p, q)$	$z = ax + by + f(a, b)$
Standard Form III	$f(z, p, q) = 0$	$u = x + ay$, $p = dz/du$, $q = a\cdot dz/du$
Standard Form IV	$f_1(x, p) = f_2(y, q)$	each side = a, solve for p, q
Charpit (general)	$f(x, y, z, p, q) = 0$	$dx/F_p = dy/F_q = dz/(pF_p + qF_q) = -dp/(F_x + pF_z) = -dq/(F_y + qF_z)$

Linear PDE with constant coefficients

Factor of $F(D, D')$	Contribution to CF
$D - mD'$	$\varphi(y + mx)$
$D - mD' - c$	$e^{cx}\,\varphi(y + mx)$
$D - mD'$ (repeated)	$\varphi_1(y + mx) + x\,\varphi_2(y + mx)$

Right-hand side $f(x, y)$	PI shortcut
$e^{ax + by}$	$f(x, y) / F(a, b)$, provided $F(a, b) \neq 0$
$\sin(ax + by)$ or $\cos(ax + by)$	replace $D^2 \to -a^2$, $D'^2 \to -b^2$, $DD' \to -ab$ in F
polynomial in x, y	expand $1/F$ as binomial in D, D'
$e^{ax+by} \cdot V(x, y)$	shift: $F(D + a, D' + b)$ acting on V

Classical PDE forms

Equation	Standard form
1-D heat	$\partial u/\partial t = c^2\, \partial^2 u/\partial x^2$
2-D heat	$\partial u/\partial t = c^2(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$
1-D wave	$\partial^2 y/\partial t^2 = c^2\, \partial^2 y/\partial x^2$
2-D Laplace	$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$
3-D Laplace	$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$

Statistics — measures of central tendency (grouped data)

Quantity	Formula
Mean	$\bar{x} = \Sigma(f \cdot x_{mid}) / \Sigma f$
Median	$L + h \cdot (N/2 - F) / f$, where L = lower limit of median class, $N = \Sigma f$, F = cumulative freq just before median class, f = freq of median class, h = class width
Mode	$L + h \cdot (f_1 - f_0) / (2f_1 - f_0 - f_2)$
Standard deviation σ	$\sqrt{[\Sigma f(x - \bar{x})^2 / \Sigma f]} = \sqrt{[\Sigma fx^2/\Sigma f - (\Sigma fx/\Sigma f)^2]}$

Correlation & regression

Quantity	Formula
Karl Pearson r	$r = \Sigma(x - \bar{x})(y - \bar{y}) / \sqrt{[\Sigma(x - \bar{x})^2 \cdot \Sigma(y - \bar{y})^2]}$
Direct (no shifting)	$r = [n \cdot \Sigma xy - \Sigma x \cdot \Sigma y] / \sqrt{[(n \cdot \Sigma x^2 - (\Sigma x)^2)(n \cdot \Sigma y^2 - (\Sigma y)^2)]}$
Regression coeff. of y on x	$b_{yx} = \Sigma(x - \bar{x})(y - \bar{y}) / \Sigma(x - \bar{x})^2 = r \cdot \sigma_y/\sigma_x$
Regression coeff. of x on y	$b_{xy} = \Sigma(x - \bar{x})(y - \bar{y}) / \Sigma(y - \bar{y})^2 = r \cdot \sigma_x/\sigma_y$
Line of y on x	$y - \bar{y} = b_{yx} (x - \bar{x})$
Line of x on y	$x - \bar{x} = b_{xy} (y - \bar{y})$
Identity	$r^2 = b_{yx} \cdot b_{xy}$; r has same sign as b_{yx} and b_{xy}

Probability distributions

Distribution	PMF / PDF	Mean	Variance
Binomial $B(n, p)$	$P(X = k) = C(n, k)\, p^k q^{n-k}$	np	npq
Poisson $P(\lambda)$	$P(X = k) = e^{-\lambda} \lambda^k / k!$	λ	λ
Normal $N(\mu, \sigma^2)$	$(1 / \sigma\sqrt{(2\pi)}) \exp[-(x - \mu)^2 / 2\sigma^2]$	μ	σ^2

Fourier transforms (Sine and Cosine)

Transform	Formula
Fourier Sine	$F_s\{f(x)\} = \sqrt{2/\pi} \int_0^\infty f(x) \sin(sx)\, dx$
Inverse Sine	$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin(sx)\, ds$
Fourier Cosine	$F_c\{f(x)\} = \sqrt{2/\pi} \int_0^\infty f(x) \cos(sx)\, dx$
Inverse Cosine	$f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos(sx)\, ds$

Many textbooks drop the $\sqrt{2/\pi}$ factor and put $2/\pi$ on one side only. Either convention is fine; just be consistent within an answer.

Part 3 — Topic-by-Topic Deep Teaching

The 12 topics below are ordered by **probability of appearing this year**. Read them in order; the later ones are wildcards.

Topic 1 — Formation of Partial Differential Equations

Probability: Near-certain. The exact question `z = f(x² + y²)` appeared in BOTH papers (Sec A Q1).

The idea

Imagine you have a function `z = f(x² + y²)`. The "f" is unknown — could be sine, could be log, could be anything. **A PDE is what you get when you eliminate that unknown function** and write down a relationship that *every* such z satisfies, no matter what f is.

It's like saying: "I don't know your exact recipe, but here's a rule every dish in your family must follow." The rule is the PDE.

For an equation with **one arbitrary function**, you typically need to **differentiate once** with respect to each of x and y, then eliminate f' (or whatever the derivative of the arbitrary function is) algebraically. For **two arbitrary constants** you do the same kind of elimination. For **two arbitrary functions**, you'll usually need second-order derivatives.

Formal procedure

Suppose **z = f(u)** where u is a known function of x, y (and possibly z). Then:

$$p = \frac{\partial z}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x}, \qquad q = \frac{\partial z}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y}.$$

Divide:

$$\frac{p}{q} = \frac{\partial u/\partial x}{\partial u/\partial y} \implies p \cdot \frac{\partial u}{\partial y} - q \cdot \frac{\partial u}{\partial x} = 0.$$

That's your PDE. The f' cancels — that's the whole trick.

Worked example (this is the PYQ)

Form the PDE from z = f(x² + y²).

Let $u = x^2 + y^2$. Then $\partial u/\partial x = 2x$ and $\partial u/\partial y = 2y$.

$$p = f'(u) \cdot 2x, \qquad q = f'(u) \cdot 2y.$$

Dividing,

$$\frac{p}{q} = \frac{2x}{2y} = \frac{x}{y} \implies py = qx \implies \boxed{py - qx = 0}.$$

> **Exam-ready text (write this verbatim for 2 marks):** Let $u = x^2 + y^2$, so that $z = f(u)$. Differentiating partially, $p = \partial z/\partial x = 2x \cdot f'(u)$ and $q = \partial z/\partial y = 2y \cdot f'(u)$. Dividing the two, $p/q = x/y$, which gives the required partial differential equation **py − qx = 0**, i.e. **yp = xq**.

Variants worth knowing

If z is given as…	Quick recipe	Resulting PDE
$z = f(x^2 + y^2)$	divide p by q	$yp - xq = 0$
$z = f(x^2 - y^2)$	divide p by q	$yp + xq = 0$
$z = f(x + y)$	$p = q$	$p - q = 0$
$z = f(x - y)$	$p = -q$	$p + q = 0$
$z = f(x) \cdot g(y)$	needs second-derivatives	$rt = ps$ (with r, s, t the 2nd-order partials)
$z = ax + by + ab$ (two constants)	eliminate a, b	$z = px + qy + pq$ (Clairaut form)

Common mistakes

- **Forgetting the chain rule.** Lots of students write $p = 2x$ directly, ignoring the $f'(u)$. Don't — you'll lose all the marks because $f'(u)$ is what *needs* to be eliminated.
- **Using the wrong p, q.** Always: $p = \partial z/\partial x$, $q = \partial z/\partial y$. Not the other way around.
- **Stopping too early.** The final answer is a relation in p, q, x, y, z with no f or f′ left.

Where it appeared

Paper A · Sec A · Q1 · 2 marks (May 2024 paper repeats it as Sec A · Q1 · 2 marks).

Topic 2 — Heat Equation and Wave Equation (definitions)

Probability: Near-certain. Both equations appear in BOTH papers, either as 1-D or 2-D.

The idea — heat equation

Think of a long thin rod being heated at one end. Temperature at every point along the rod changes over time. The **rate at which temperature changes at a point** is proportional to **how curved the temperature profile is** at that point — that curvature is the second spatial derivative.

If the curve dips down at a point (concave up), heat is flowing in from both neighbours, so the temperature rises. If the curve bumps up (concave down), heat flows out, so temperature drops. That's exactly what $\partial u/\partial t = c^2 \partial^2 u/\partial x^2$ says.

The constant **c^2** is the *thermal diffusivity*: bigger c^2 means the rod equalises temperature faster.

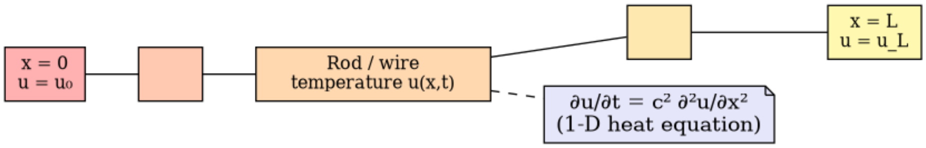

Figure 1: Heat equation in a rod

The idea — wave equation

Now imagine a stretched guitar string, slightly plucked. Each tiny piece of the string feels a restoring force from its neighbours — proportional to how curved the string is at that point. By Newton's second law (force $\propto$ acceleration), the **acceleration** of each piece (which is $\partial^2 y/\partial t^2$) is proportional to the **curvature** ($\partial^2 y/\partial x^2$).

So $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$ — same shape as the heat equation but with two t-derivatives. That second time derivative is what gives waves their oscillation (and lets them carry energy back and forth).

The constant $c^2 = T/\rho$ where T is tension and ρ is linear mass density. Pluck a tighter string — c grows, waves move faster.

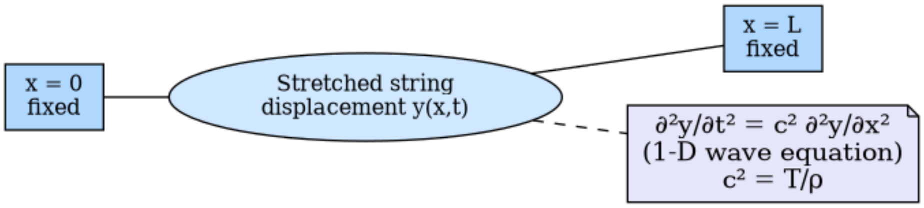

Figure 2: Wave equation in a stretched string

Formal statements

Phenomenon	Equation
1-D heat equation	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
2-D heat equation	$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
3-D heat equation	$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$
1-D wave equation	$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$
2-D wave equation	$\frac{\partial^2 u}{\partial t^2} = c^2 (u_{xx} + u_{yy})$
Laplace equation (steady-state)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Exam-ready text — 1-D heat: The one-dimensional heat (or diffusion) equation is $\mathbf{\partial u/\partial t = c^2 \cdot \partial^2 u/\partial x^2}$, where $u(x, t)$ is the temperature at position x and time t and c^2 is the thermal diffusivity of the medium.

Exam-ready text — wave: The one-dimensional wave equation is $\mathbf{\partial^2 y/\partial t^2 = c^2 \cdot \partial^2 y/\partial x^2}$, where $y(x, t)$ is the displacement of the string at position x and time t, and $c^2 = T/\rho$ with T = tension and ρ = mass per unit length.

Exam-ready text — 2-D heat: The two-dimensional heat equation governing the flow of heat in a plate is $\mathbf{\partial u/\partial t = c^2 \cdot (\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)}$ where $u(x, y, t)$ is the temperature at point (x, y) at time t.

Common mistakes

- Mixing up: heat has **one** time derivative, wave has **two**. Heat smooths things out (irreversible); wave preserves shapes (reversible).
- Forgetting **c^2** — many students write `∂u/∂t = ∂²u/∂x²`. The c^2 carries the physical units and is worth a mark.
- For 2-D, writing only `c²·∂²u/∂x²` and forgetting the y-piece.

Where it appeared

Heat eq: Paper A Sec A Q2 (2-D) · 2 marks · Paper B Sec A Q2 (1-D) · 2 marks
Wave eq: Paper A Sec A Q3 · 2 marks · Paper B Sec A Q3 · 2 marks

Topic 3 — Non-linear PDE of First Order: Standard Forms

Probability: Near-certain. Pq=1 in Paper A and $p^2 + q^2 = 1$ in Paper B (both Section B, 9 marks).

The idea

A first-order PDE in two variables is **non-linear** when p or q appear in powers, products, or non-linear functions (like sin p, e^q, pq, p^2). The general theory uses Charpit's method (Topic 7), but a lot of "common" non-linear PDEs fall into one of **four standard forms** where a smart substitution gives the answer instantly.

The key insight: for Form I (when only p and q appear, no x, y, z), the simplest possible solution is a plane:

$$z = ax + by + c.$$

Why? Because for a plane, p = a (constant) and q = b (constant), so substituting into f(p, q) = 0 forces a relation between a and b. That's the **complete integral**.

The other three forms are clever variations of the same idea.

Standard Form I — f(p, q) = 0

Recipe: Try z = ax + by + c. Then p = a, q = b. Substituting into f(p, q) = 0 gives f(a, b) = 0, which fixes one of a, b in terms of the other. The result is the complete integral with two arbitrary constants (one of a, b after the relation, and c).

Example — Pq = 1 (Paper A Q6):

Set z = ax + by + c. Then ab = 1 ⇒ b = 1/a.

$$\boxed{z = ax + \frac{y}{a} + c}$$

Example — $p^2 + q^2 = 1$ (Paper B Q6):

Set z = ax + by + c. Then $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$.

$$\boxed{z = ax + y\sqrt{1 - a^2} + c}$$

Standard Form II (Clairaut's form) — z = px + qy + f(p, q)

Recipe: Replace p with a, q with b directly. Complete integral:

$$z = ax + by + f(a, b).$$

This is the *direct* analogue of the ODE Clairaut form y = px + f(p).

Example: z = px + qy + pq has complete integral z = ax + by + ab.

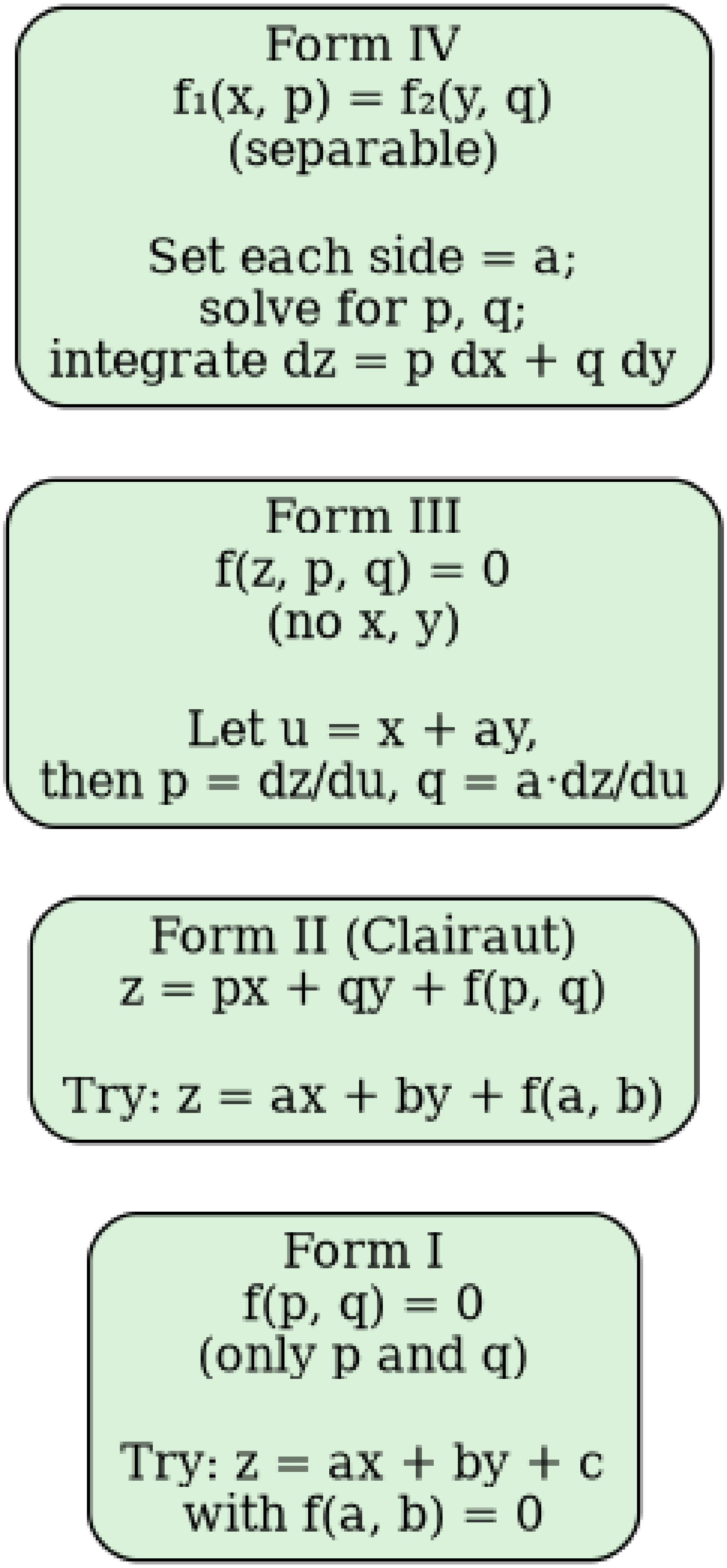

Figure 3: Standard forms of first-order non-linear PDE

Standard Form III — f(z, p, q) = 0 (no x or y)

Recipe: Let u = x + ay (a is an arbitrary constant). Treat z as a function of u alone. Then

$$p = \frac{dz}{du}, \qquad q = a\,\frac{dz}{du}.$$

Substitute back into f(z, p, q) = 0; you get an ODE in z and dz/du which is separable. Integrate.

Example: z = pq. Let u = x + ay so p = dz/du, q = a·dz/du. Then $z = a\,(dz/du)^2 \Rightarrow dz/du = \sqrt{z/a} \Rightarrow$ separable.

Standard Form IV — $f_1(x, p) = f_2(y, q)$

Recipe: Each side is a function only of one variable. Set both sides equal to a constant a:

$$f_1(x,p) = a \;\text{ and }\; f_2(y,q) = a.$$

Solve each for p and q respectively. Then integrate dz = p dx + q dy.

Example: $p^2 + q^2 = x + y$. Rewrite as $p^2 - x = y - q^2$. The left side is a function of (x, p) only, right of (y, q) only. So $p^2 - x = a$ and $y - q^2 = a$. Then $p = \sqrt{x + a}$, $q = \sqrt{y - a}$. Integrate.

How to recognise which form

What you see	Which form?
Only p and q (no x, y, z)	Form I
z = px + qy + something(p, q)	Form II
Only z, p, q (no x, y)	Form III
One side has only (x, p), other has only (y, q)	Form IV
None of the above	Use **Charpit**

Common mistakes

- Trying z = ax + by + c for Form III — it doesn't work because the PDE has z in it, so a plane won't generally satisfy it.
- Forgetting the constant c in Form I. The complete integral of an n-variable PDE needs n arbitrary constants.
- In Form III, students often write `p = dz/du` and `q = dz/du` (missing the factor a). Be careful.

Where it appeared

Paper A Sec B Q6 (Pq = 1) · 9 marks Paper B Sec B Q6 ($p^2 + q^2 = 1$) · 9 marks

Topic 4 — Linear PDE with Constant Coefficients

Probability: Near-certain. (D+1)(D+D’+1)z = sin(x+2y) in A and (D+1)(D+D’−1)z = sin(x+2y) in B. Section C, 14 marks each.

The idea

We’re treating partial differentiation as if it were algebra. Define:

$$D = \frac{\partial}{\partial x}, \qquad D' = \frac{\partial}{\partial y}.$$

Then a PDE like `(D + 1)(D + D' + 1) z = sin(x + 2y)` is just `F(D, D') z = f(x, y)` where F is a *polynomial* in two operators that commute. We solve it exactly like an ODE:

$$z = \underbrace{\text{Complementary Function}}_{\text{solves } F(D,D')z=0} + \underbrace{\text{Particular Integral}}_{F(D,D')z=f(x,y)}$$

Building the CF — factor F(D, D’)

Rule: For every linear factor of the form `(D − mD' − c)`, the corresponding piece of the CF is

$$e^{c\,x}\,\phi(y + m\,x),$$

where φ is an **arbitrary function**. Stack the pieces for each factor.

A few special cases:

Factor	CF piece
`D − mD'` (c = 0)	φ(y + mx)
`D + 1` (i.e. D − 0·D’ − (−1), so m=0, c=−1)	e^{−x} φ(y)
`D + D' + 1` (i.e. D − (−1)D’ − (−1), so m=−1, c=−1)	e^{−x} φ(y − x)
`D + D' − 1` (i.e. D − (−1)D’ − 1, so m=−1, c=1)	e^{x} φ(y − x)
`(D − mD')²` (repeated)	$\varphi_1(y + mx) + x\cdot\varphi_2(y + mx)$

Building the PI — three useful shortcuts

The PI is `z_p = f(x, y) / F(D, D')`. The shortcut you use depends on the shape of f.

Shortcut 1 — exponential right-hand side

If f(x, y) = e^{ax + by}, then

$$\text{PI} = \frac{e^{ax+by}}{F(a,b)}, \quad \text{provided } F(a,b) \neq 0.$$

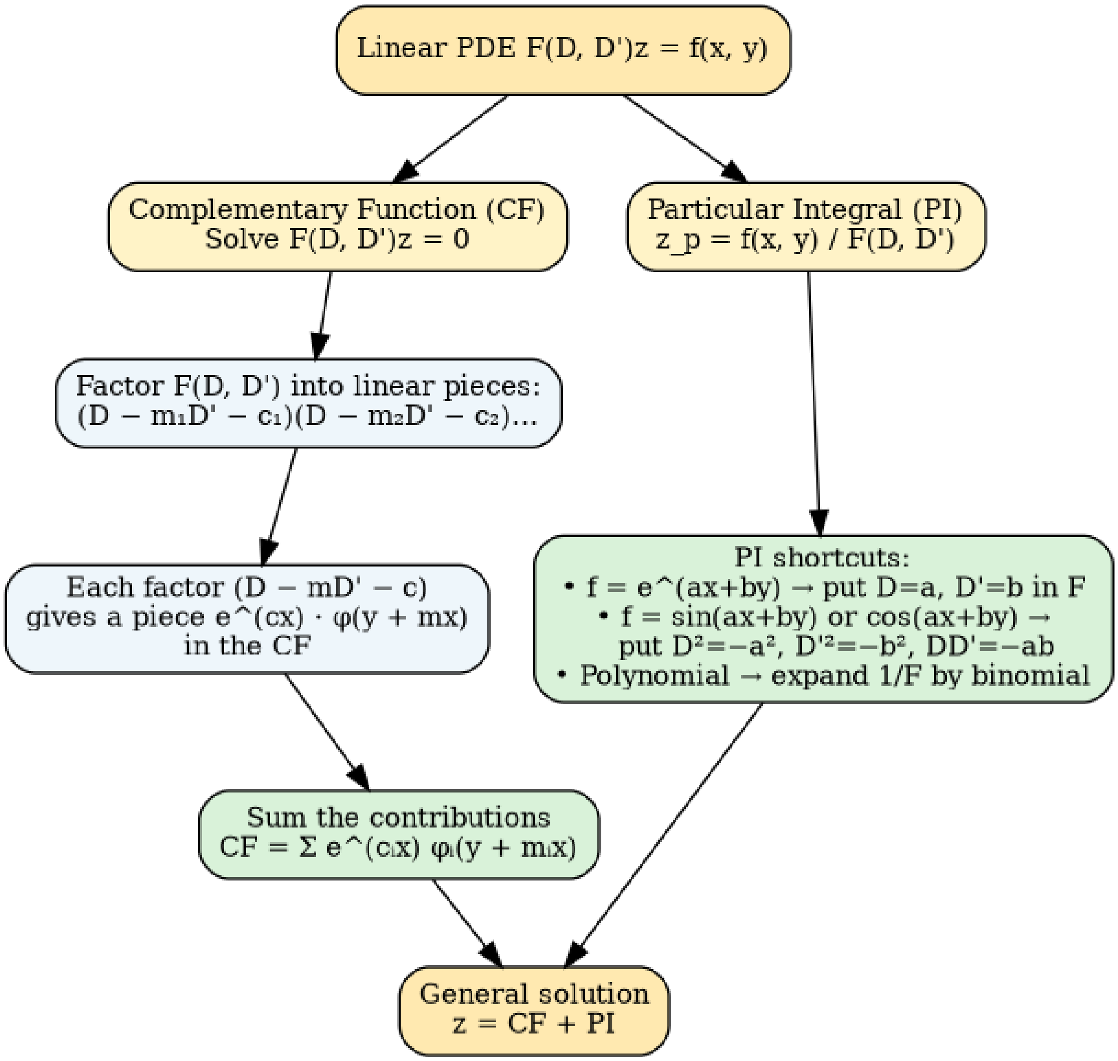

Figure 4: Operator method

Shortcut 2 — sine or cosine right-hand side

If f(x, y) = sin(ax + by) or cos(ax + by), substitute

$$D^2 \to -a^2, \quad D'^2 \to -b^2, \quad DD' \to -ab$$

inside F. **Note:** Single D and D' (not squared) remain as differentiation operators; you may still have to apply them separately. The simplest trick: write z_p = A sin(ax + by) + B cos(ax + by) and **substitute into the original PDE** to find A and B.

Shortcut 3 — polynomial right-hand side

Expand `1/F(D, D')` as a binomial series in D and D', keeping only enough terms to kill the polynomial.

Why the substitution trick for sin/cos

When you apply D^2 to sin(ax+by), you get $-a^2$ sin(ax+by) — the function comes back multiplied by $-a^2$. So D^2 *acts like* the number $-a^2$ on this specific function. Same for D'^2 and DD'. That's why we just replace.

But **single D** or **single D'** turns sin into cos (and vice versa), so it doesn't reduce to a clean multiplier. When F contains odd-degree terms in D or D' separately, the cleanest approach is the "assume the form A sin + B cos and substitute" method.

Worked example (Paper A Q9, full solution in Part 4)

We'll solve `(D + 1)(D + D' + 1) z = sin(x + 2y)` step-by-step in Part 4. Spoiler: CF has two e^{−x} pieces, and the PI is found by the "A sin + B cos" trick because the operator has odd-degree terms.

Common mistakes

- Trying to apply the sin/cos shortcut "blindly" when the operator has single-D or single-D' terms. **Always check** that what you get back is consistent — the safest move is to verify by substitution.
- Dropping the e^{cx} prefactor when the linear factor has a constant term. (D + 1) gives e^{−x} φ(y), not just φ(y).
- Confusing m and c: for `(D + α + βD')`, write it as `D − (−β)D' − (−α)`, so m = −β, c = −α.
- Forgetting that φ is *arbitrary* — it should be left as φ in the answer (or labelled $φ_1$, $φ_2$ if there are multiple factors). Don't try to pin down what φ is.

Where it appeared

Paper A Sec C Q9 · 14 marks: `(D + 1)(D + D' + 1) z = sin(x + 2y)` Paper B Sec C Q9 · 14 marks: `(D + 1)(D + D' − 1) z = sin(x + 2y)`

Topic 5 — Charpit's Method

Probability: Near-certain. Both papers have it in Section C, 14 marks.

The idea

When a non-linear first-order PDE *doesn't* fit any standard form, you need a general technique. **Charpit's method** is that technique. It says: "find any *second* PDE that's *compatible* with the original one (i.e., they share solutions), then solve the two together for p and q. Integrate dz = p dx + q dy."

The compatibility condition is a system of five "auxiliary equations" — Charpit's equations — and you only need *one* manageable pair from them.

The five auxiliary equations

Given F(x, y, z, p, q) = 0, Charpit's equations are:

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{p\,F_p + q\,F_q} = \frac{-dp}{F_x + p\,F_z} = \frac{-dq}{F_y + q\,F_z}.$$

You don't have to use all five. Pick whichever pair gives the cleanest relation between two of {x, y, z, p, q}. Often the last two — dp/(stuff) and dq/(stuff) — give the cleanest answer.

Step-by-step recipe

1. **Write F = 0.** Move everything to one side.
2. **Compute the five partial derivatives** of F: F_x, F_y, F_z, F_p, F_q.
3. **Write Charpit's equations.**
4. **Pick the simplest pair** (often −dp/(F_x + pF_z) = −dq/(F_y + qF_z)). Sometimes the cleanest is dp/p or dq/q.
5. **Integrate** that pair to get a second equation, typically involving p, q and a constant a.
6. **Solve the two equations together** (the original F = 0 and your new equation) for p and q in terms of x, y, z and a.
7. **Integrate** dz = p dx + q dy. You'll pick up a second constant b.

Worked example (Paper A Q10 — full solution in Part 4)

We'll solve px + qy = pq end-to-end. The clean pair turns out to be −dp/p = −dq/q, which integrates to p = aq. From there the algebra is mechanical.

Common mistakes

- **Mis-signing F_x and F_y.** Charpit's equations have *negative* numerators for dp and dq. Lots of marks get lost here.
- Picking the wrong pair and getting stuck in algebra. Try the **dp, dq pair first** — it often works.

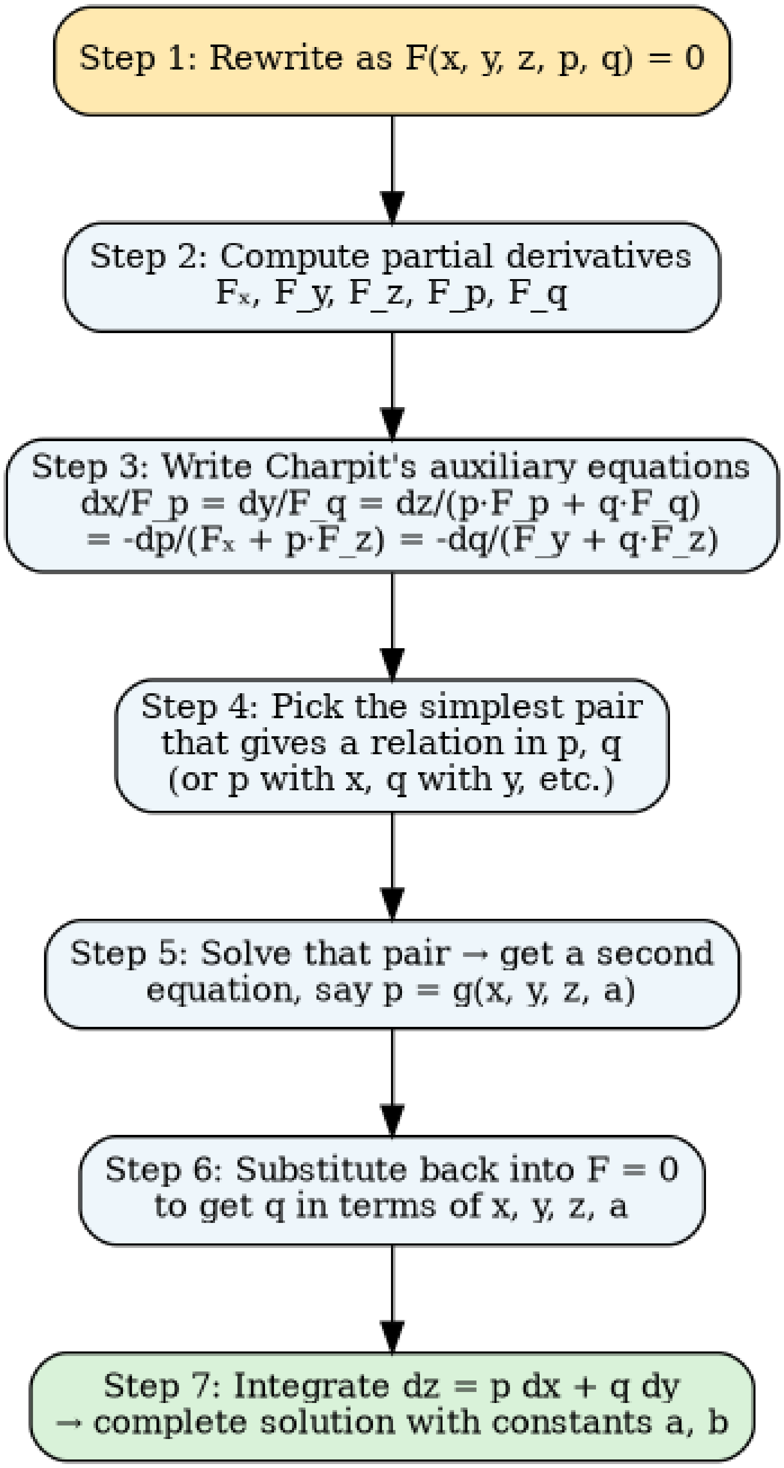

Figure 5: Charpit’s workflow

- Forgetting that F_z multiplies p (or q) in the numerator. If F doesn’t depend on z, then F_z = 0 and the equations simplify.
- Not getting *both* constants. A complete integral has two arbitrary constants for a first-order PDE in two variables.

Where it appeared

Paper A Sec C Q10 · 14 marks: `Solve by Charpit's method: px + qy = pq`
Paper B Sec C Q10 · 14 marks: `Solve by Charpit's method: (p² + q²)y = qz`

Topic 6 — Mode of Grouped Data

Probability: Near-certain. The same age-frequency table style appeared in BOTH papers (Sec C, 14 marks). Mode formula is on the must-know list.

The idea

Mode is the value that appears **most often** in a dataset. For raw numbers, you just look. For *grouped* (continuous) data, you can't see individual values — you only have classes. So you do two things:

1. **Identify the modal class** — the class with the highest frequency.
2. **Refine within that class** — assume the data is unevenly distributed even *inside* the modal class, with more density on whichever side the frequency neighbours are higher.

The standard formula does exactly that.

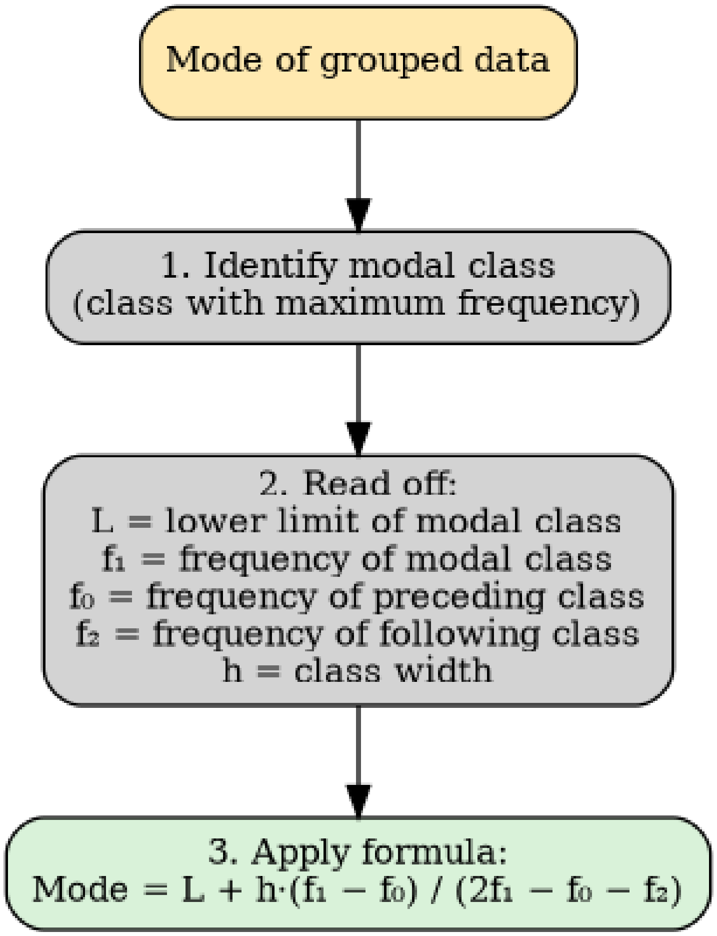

Figure 6: Mode procedure

The formula

$$\text{Mode} \;=\; L \;+\; h \cdot \frac{f_1 - f_0}{2f_1 - f_0 - f_2}.$$

Symbol	Meaning
L	Lower limit of the modal class
f_1	Frequency of the modal class
f_0	Frequency of the class *just before* the modal class
f_2	Frequency of the class *just after* the modal class
h	Class width (size of the class interval)

Worked example (Paper A Q11)

Data:

Age	0–6	6–12	12–18	18–24	24–30
Frequency	7	10	24	**36**	23

The modal class is **18–24** (highest frequency 36). So L = 18, $f_1 = 36$, $f_0 = 24$, $f_2 = 23$, h = 6.

$$\text{Mode} = 18 + 6 \cdot \frac{36 - 24}{2(36) - 24 - 23} = 18 + 6 \cdot \frac{12}{25} = 18 + 2.88 = \boxed{20.88}.$$

Common mistakes

- **Choosing the wrong modal class.** Always pick the class with the **largest frequency**, not “the middle” or “the one with the widest range”.
- **Using cumulative frequencies** — that’s median, not mode.
- **Confusing f_0 and f_2.** f_0 is *before* the modal class (smaller class boundary), f_2 is *after*.
- **Forgetting h** — you must multiply by the class width.

Where it appeared

Paper A Sec C Q11 · 14 marks Paper B Sec C Q11 · 14 marks

Topic 7 — Mean of Grouped Data (with comparison to median)

Probability: Near-certain (both papers have it, Sec B 9 marks). Median is the natural rotation candidate this year.

The idea — mean

For a grouped frequency distribution, the “mean” is the **weighted average of class midpoints**, weighted by the class frequencies. Each midpoint represents its class as a single value, and frequency tells you how many actual observations sit in that class.

The formula

$$\bar{x} \;=\; \frac{\sum f_i\, x_i}{\sum f_i},$$

where x_i = midpoint of class i and f_i = frequency of class i.

For larger numbers, use the **shortcut (assumed mean) method**:

$$\bar{x} \;=\; A + h \cdot \frac{\sum f_i\, d_i}{\sum f_i}, \quad d_i = \frac{x_i - A}{h}.$$

Here A is any convenient class midpoint, and `d_i` are integer “deviations”.

Worked example (Paper A Q7)

Data:

Class	0–10	10–20	20–30	30–40
Frequency	7	8	15	10

Midpoints: 5, 15, 25, 35. Compute f·x:

Class	f	x	f·x
0–10	7	5	35
10–20	8	15	120
20–30	15	25	375
30–40	10	35	350
Σ	**40**		**880**

$$\bar{x} = \frac{880}{40} = \boxed{22}.$$

Median (the rotation candidate for this year)

If the next paper rotates mean → median, here’s the formula:

$$\text{Median} = L + h \cdot \frac{(N/2) - F}{f},$$

where L = lower limit of the median class, N = Σf, F = cumulative frequency *just before* the median class, f = frequency of the median class, h = class width. The **median class** is the class containing the (N/2)-th observation in the cumulative frequencies.

Common mistakes

- Forgetting to use class **midpoints** (using lower limits or upper limits instead).
- Arithmetic slips when adding fx — show the table.
- For median: counting cumulative frequencies from the wrong side.

Where it appeared

Paper A Sec B Q7 · 9 marks (mean) Paper B Sec B Q8 · 9 marks (mean)

Topic 8 — Correlation Coefficient and Regression Lines

Probability: Near-certain. Paper A Q13 (correlation + regression) and Paper B Q12 (correlation). Both 14 marks.

The idea — correlation

Correlation answers: **“As x increases, does y tend to increase, decrease, or do nothing?”** The Karl Pearson coefficient r is a number in $[-1, +1]$:

- $r = +1$: perfect positive — every increase in x is matched by a proportional increase in y, points lie on an upward straight line.
- $r = 0$: no linear relation — *but they could still be related non-linearly!*
- $r = -1$: perfect negative — every increase in x matched by a proportional decrease in y.

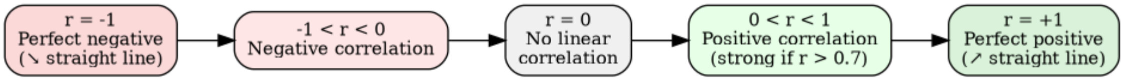

Figure 7: Correlation scale

The formula

The cleanest computational form (no shifting needed):

$$r \;=\; \frac{n\sum xy - \sum x \sum y}{\sqrt{\left[\,n\sum x^2 - (\sum x)^2\,\right]\left[\,n\sum y^2 - (\sum y)^2\,\right]}}.$$

If means are nice integers, the **shifted form** is faster. Let $X = x - \bar{x}$, $Y = y - \bar{y}$:

$$r \;=\; \frac{\sum XY}{\sqrt{\sum X^2 \cdot \sum Y^2}}.$$

The idea — regression

Once you know x and y are correlated, you might want to **predict y for a given x** (or vice versa). The **regression line of y on x** is the line that minimises the sum of squared *vertical* residuals (errors in predicting y). The line of x on y minimises squared *horizontal* residuals.

There are two different lines because predicting y from x is not the same problem as predicting x from y.

Regression formulas

$$\text{Line of } y \text{ on } x: \quad y - \bar{y} = b_{yx}\,(x - \bar{x}), \quad b_{yx} = \frac{\sum XY}{\sum X^2} = r\,\frac{\sigma_y}{\sigma_x}.$$

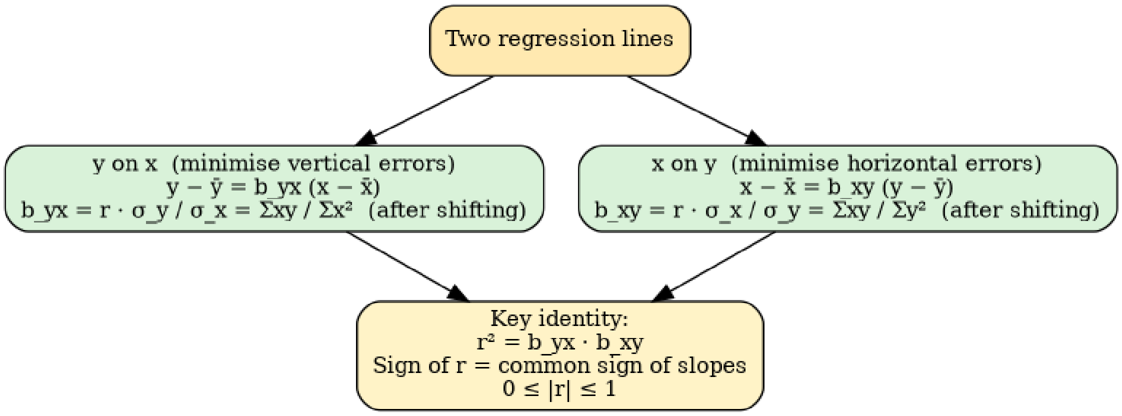

Figure 8: Two regression lines

$$\text{Line of } x \text{ on } y: \quad x - \bar{x} = b_{xy}\,(y - \bar{y}), \quad b_{xy} = \frac{\sum XY}{\sum Y^2} = r\,\frac{\sigma_x}{\sigma_y}.$$

Identity: $r^2 = b_{yx} \cdot b_{xy}$. So if you compute the two slopes separately you can recover $|r|$ as the geometric mean.

The two regression lines always **intersect at the point $(\bar{x}, \bar{y})$** — the centre of mass. If $r = \pm 1$, the two lines coincide; if $r = 0$, they are perpendicular (one horizontal, one vertical).

Worked example (this is the PYQ; full solution in Part 4)

For $x = 1 \ldots 9$ and the corresponding y values in Paper A, we'll see $r = 0.95$, with both regression lines $y = 0.95x + 7.25$ and $x = 0.95y - 6.4$. Almost perfectly correlated.

Common mistakes

- **Squaring wrong.** The denominator uses $(\Sigma x)^2$ in the direct formula, **not** Σx^2 squared again. Be careful with brackets.
- Using $r \cdot \sigma_y/\sigma_x$ when you haven't computed σ_y or σ_x. The $\Sigma XY / \Sigma X^2$ form is usually faster.
- Forgetting that the line of y on x and x on y are different — you'll lose marks if you give the same line twice.
- Reporting $r \geq 1$ or $r \leq -1$ — that's a sign you've made an arithmetic error.

Where it appeared

Paper A Sec C Q13 · 14 marks: 9 pairs of (x, y), find r and both regression lines.
Paper B Sec C Q12 · 14 marks: 7 pairs of (x, y), find r.

Topic 9 — Binomial Distribution

Probability: Near-certain. Bomb-strike problem in Paper A and mean/variance problem in Paper B (both Section C, 14 marks). Plus first-moment problem in Paper B Section B.

The idea

You flip a coin n times. Each flip is independent and has the same probability p of “success” (whatever you decide success is). Let X = total number of successes across all n flips. Then **X follows a Binomial distribution**, written X ~ B(n, p).

The PMF gives the probability of exactly k successes out of n trials:

$$P(X = k) \;=\; \binom{n}{k}\, p^k\, q^{n-k}, \qquad q = 1 - p.$$

Where the formula comes from: there are C(n, k) ways to choose *which* k trials are successes. Each specific sequence has probability p^k · q^(n−k). Multiply.

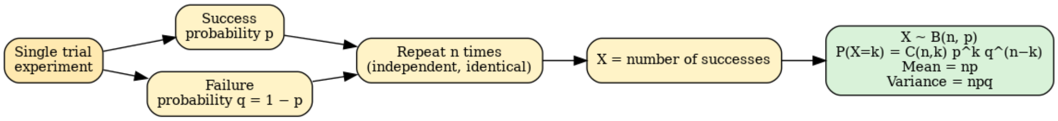

Figure 9: Binomial setup

Key moments

Quantity	Value	Why
Mean (E[X])	np	n trials, each contributing p on average
Variance	npq	n independent contributions of pq each
Standard deviation	√(npq)	square root of variance
First moment about origin	$\mu_1' = np$	same as mean
Second moment about origin	$\mu_2' = np(np + q) = (np)^2 + npq$	μ_2' = mean² + var

“First moment about the origin” — Paper B Q7

The r-th moment about the origin is `μ_r' = E[X^r]`. So the first moment is just the mean.

For Binomial:

$$\mu_1' = E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}.$$

Pull out one factor of np from each term (using the identity `k · C(n,k) = n · C(n−1, k−1)`):

$$\mu_1' = np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}q^{n-k} = np\cdot(p+q)^{n-1} = np\cdot 1 = \boxed{np}.$$

(That clean cancellation by the binomial theorem is why “Binomial” is named that.)

When you’re given mean and variance, find n and p

If mean = np and variance = npq are both given:

$$q = \frac{\text{variance}}{\text{mean}}, \qquad p = 1-q, \qquad n = \frac{\text{mean}}{p}.$$

Example (Paper B Q13): mean = 4, variance = 2. So q = 2/4 = 1/2 ⇒ p = 1/2 ⇒ n = 4/(1/2) = **8**. Now use B(8, 1/2) to compute the probabilities asked.

Common mistakes

- Confusing p and q. **p is the probability of success per trial, q = 1 − p is failure.**
- Forgetting C(n, k) in the formula.
- Not reading the question carefully: “exactly 2”, “less than 2”, “at least 2” — those are three different sums.
- For “at least k”, use **complement**: P(X ≥ k) = 1 − [P(X = 0) + P(X = 1) + … + P(X = k−1)]. It’s almost always less computation than summing from k to n.

Where it appeared

> Paper A Sec C Q12 · 14 marks: 6 bombs, hit prob 1/3, find P(exactly 2) and P(at least 2). Paper B Sec C Q13 · 14 marks: mean = 4, var = 2, find P(exactly 2), P(less than 2), P(at least 2). Paper B Sec B Q7 · 9 marks: first moment about origin for Binomial — derive np.

Topic 10 — Simple Classical Probability

Probability: Near-certain. Both papers have a Section A 2-mark probability question.

The idea

When every outcome is **equally likely** (like a fair die or coin), the probability of an event E is just:

$$P(E) = \frac{\#\text{ favourable outcomes}}{\#\text{ total outcomes}}.$$

That's the whole thing. The hard part is counting *correctly* — that's where combinations come in for problems like "drawing balls from an urn".

Examples (the two PYQs)

Paper A Q5: Probability of throwing an odd number with an ordinary six-faced die.

Total outcomes = 6 (the faces 1, 2, 3, 4, 5, 6). Favourable outcomes (odd) = {1, 3, 5} = 3.

$$P(\text{odd}) = \frac{3}{6} = \boxed{\frac{1}{2}}.$$

Paper B Q5: Probability of getting 2 heads in 3 tosses of a fair coin.

This is actually a Binomial mini-problem: n = 3, p = 1/2, find P(X = 2).

$$P(X = 2) = \binom{3}{2}(1/2)^2\,(1/2)^1 = 3 \cdot \frac{1}{8} = \boxed{\frac{3}{8}}.$$

Alternatively, by listing: total outcomes = 2^3 = 8 sequences. The ones with exactly two heads are HHT, HTH, THH — that's 3. So P = 3/8.

Urn / combination probability (Paper A Q8 sneaks this in)

An urn has 10 black and 10 white balls. P(two balls drawn are the same colour)?

Total ways to pick 2 from 20: C(20, 2) = 190.

Outcome	Count
Both black	C(10, 2) = 45
Both white	C(10, 2) = 45
Same colour total	45 + 45 = **90**

$$P(\text{same colour}) = \frac{90}{190} = \boxed{\frac{9}{19}}.$$

Other tiny problems to be ready for

Question	Quick answer
P(getting a sum of 7 on two dice)	6/36 = 1/6 (count (1,6), (2,5),…, (6,1))
P(drawing a king from a pack of 52 cards)	4/52 = 1/13
P(drawing a king OR a heart)	4/52 + 13/52 − 1/52 = 16/52 (inclusion-exclusion)
P(2 heads in 2 tosses)	1/4
P(at least one head in 3 tosses)	$1 - (1/2)^3 = 7/8$

Common mistakes

- Forgetting that “exactly 2 heads in 3 tosses” needs **C(3, 2)** because the heads can occur in different positions.
- Adding probabilities of mutually exclusive events: P(A or B) = P(A) + P(B) only when they can’t happen together.
- Mis-counting “same colour” as just “both white”. Read carefully.

Where it appeared

Paper A Sec A Q5 · 2 marks (die) Paper B Sec A Q5 · 2 marks (coins) Paper A Sec B Q8 · 9 marks (urn)

Topic 11 — Fourier Sine and Cosine Transforms

Probability: High (definition slot). Sine appeared in Paper A Sec A Q4; Cosine is the natural rotation candidate.

The idea

A **Fourier transform** is the integral analogue of writing a function as a sum of sines and cosines. For functions defined on `[0, ∞)`, two specialised versions are popular:

- **Sine transform** — used when boundary conditions involve the *function value* itself (e.g. u(0, t) = 0).
- **Cosine transform** — used when boundary conditions involve the *derivative* (e.g. u_x(0, t) = 0).

The choice is forced by *which derivative term* appears in the integration by parts when you transform a PDE.

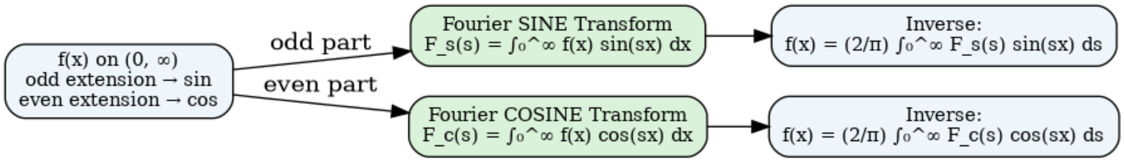

Figure 10: Fourier sine vs cosine

The formulas (definition slot)

Quantity	Formula
Fourier Sine Transform	$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \sin(sx)\, dx$
Inverse Fourier Sine Transform	$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s)\ \sin(sx)\, ds$
Fourier Cosine Transform	$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos(sx)\, dx$
Inverse Fourier Cosine Transform	$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s)\ \cos(sx)\, ds$

> **Exam-ready text (Fourier Sine Transform, 2 marks):** The Fourier Sine Transform of a function f(x) defined on (0, ∞) is **F_s{f(x)} = √(2/π) ∫₀^∞ f(x) sin(sx) dx**. Its inverse is **f(x) = √(2/π) ∫₀^∞ F_s(s) sin(sx) ds**.

Convention warning

Some textbooks omit the √(2/π) factor in the forward transform and put 2/π in the inverse. Both conventions are correct as long as you're consistent. In the exam, write the symmetric version above unless your textbook used a different one — and **always state the convention** you're using if it differs.

Where it appeared

Paper A Sec A Q4 · 2 marks

Topic 12 — Lagrange's Linear PDE

Probability: High (one paper had the definition; the *solving* version is overdue).

The idea

A **Lagrange's linear equation** is a first-order linear PDE in the form

$$Pp + Qq = R,$$

where P, Q, R are functions of x, y, z (but **not** of p or q). The method of solving it: write the **auxiliary (or subsidiary) equations**

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

This is a system of two ordinary equations in three unknowns x, y, z. Find **any two independent integrals** of this system — call them $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. The general solution to the PDE is then

$$\phi(u, v) = 0,$$

with φ an arbitrary function.

Why this works (one-line intuition)

The auxiliary equations describe **curves on which the PDE is automatically satisfied** (the "characteristic curves"). Any function constant along these curves gives a valid solution surface.

Step-by-step recipe

1. Read off P, Q, R from `Pp + Qq = R`.
2. Write `dx/P = dy/Q = dz/R`.
3. Find two independent first integrals (try simple pairs: `dx/P = dy/Q`; or use a multiplier — e.g., add up using clever ratios).
4. Call them $u = c_1$ and $v = c_2$.
5. Write the general solution as φ(u, v) = 0 or equivalently u = ψ(v).

Worked mini-example

Solve `p − q = z`.

Here P = 1, Q = −1, R = z. Auxiliary: dx/1 = dy/(−1) = dz/z.

- From dx/1 = dy/(−1): $x + y = c_1$. ✓
- From dx/1 = dz/z: dz/z = dx ⇒ $\ln z = x + \ln c_2$ ⇒ $z e^{-x} = c_2$. ✓

General solution: $\varphi(x + y, z \cdot e^{-x}) = 0$.

Exam-ready text (definition slot, 2 marks): Lagrange's linear partial differential equation is the first-order linear PDE of the form **$Pp + Qq = R$**, where $p = \partial z/\partial x$, $q = \partial z/\partial y$, and P, Q, R are functions of x, y, z. Its general solution is found by solving the auxiliary equations **$dx/P = dy/Q = dz/R$** to obtain two independent integrals $u = c_1$ and $v = c_2$, giving the solution $\varphi(u, v) = 0$.

Common mistakes

- Trying to use Lagrange when there's a pq or p^2 term — that's nonlinear, you need Charpit or a standard form.
- Picking dependent integrals (e.g. u and 2u both shifted). The two integrals must be **functionally independent**.
- Forgetting that φ is *arbitrary* in the final answer.

Where it appeared

Paper B Sec A Q4 · 2 marks (definition)

Part 4 — Complete Solutions to Every PYQ

Each solution is given in the *exact* style and depth you should reproduce in the answer book. For 2-mark questions the answers are short; for 14-mark questions they're worked through with intermediate algebra so you can copy the structure.

Paper A — Earlier Session

Section A (5 × 2 = 10) — attempt all

Q1 (2 marks) · Form the partial differential equation from z = f(x² + y²).

Let u = x² + y², so z = f(u). Differentiating partially:

$$p = \frac{\partial z}{\partial x} = f'(u) \cdot 2x, \qquad q = \frac{\partial z}{\partial y} = f'(u) \cdot 2y.$$

Dividing:

$$\frac{p}{q} = \frac{2x}{2y} = \frac{x}{y} \implies py - qx = 0.$$

Required PDE: yp − xq = 0.

Q2 (2 marks) · Write the two-dimensional heat equation.

The two-dimensional heat (or diffusion) equation is

$$\boxed{\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)}$$

where u(x, y, t) is the temperature at the point (x, y) in the plane at time t, and c² is the thermal diffusivity of the medium.

Q3 (2 marks) · Write the wave equation.

The one-dimensional wave equation is

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

where y(x, t) is the displacement of a vibrating string at position x and time t, and c² = T/ρ with T = tension and ρ = mass per unit length.

Q4 (2 marks) · Write Fourier Sine Transform.

The Fourier Sine Transform of a function f(x) defined on (0, ∞) is

$$\boxed{F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, \sin(sx)\, dx}$$

and the corresponding inverse transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s)\ \mathsf{sin}(sx)\, ds.$$

Q5 (2 marks) · Find the probability of throwing an odd number with an ordinary six-faced die.

A six-faced die has six equally likely outcomes {1, 2, 3, 4, 5, 6}. The odd outcomes are {1, 3, 5}, three in number.

$$P(\mathsf{odd}) = \frac{3}{6} = \boxed{\tfrac{1}{2}}.$$

Section B (any 2 of 3 × 9 = 18)

Q6 (9 marks) · Solve Pq = 1.

This is a non-linear first-order PDE of the form `f(p, q) = 0` (Standard Form I), since the equation involves only p and q.

Step 1 — Try a planar complete integral. Assume the form

$$z = ax + by + c.$$

For this, p = a and q = b.

Step 2 — Substitute. The PDE pq = 1 becomes

$$ab = 1 \implies b = \frac{1}{a}.$$

Step 3 — Write the complete integral.

$$\boxed{z = a\,x + \frac{y}{a} + c}$$

This is the complete integral with two arbitrary constants a and c.

Step 4 (optional) — Singular and general integrals. The singular integral comes from eliminating a and c between $z = ax + y/a + c$, $\partial z/\partial a = x - y/a^2 = 0$, and $\partial z/\partial c = 1 \neq 0$. The condition $\partial z/\partial c = 1 \neq 0$ means no singular integral exists. For the general integral, take $c = \varphi(a)$ and eliminate a between $z = ax + y/a + \varphi(a)$ and $\partial z/\partial a = x - y/a^2 + \varphi'(a) = 0$.

> **Key takeaway:** When you see only p and q in a PDE, jump straight to z = ax + by + c. The PDE then becomes a relation between a and b — solve it, plug back, done.

Q7 (9 marks) · Find the arithmetic mean of the distribution

Class	0–10	10–20	20–30	30–40
Frequency	7	8	15	10

Step 1 — Tabulate midpoints (x) and compute f·x.

Class	Midpoint x	Frequency f	f·x
0–10	5	7	35
10–20	15	8	120
20–30	25	15	375
30–40	35	10	350
Total		**Σf = 40**	**Σfx = 880**

Step 2 — Apply the mean formula.

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{880}{40} = \boxed{22}.$$

Key takeaway: Build the table, compute the column-totals, divide. Always show the table — examiners give partial credit even if your final arithmetic is off.

Q8 (9 marks) · An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour.

Step 1 — Total ways to draw 2 balls from 20.

$$\binom{20}{2} = \frac{20 \cdot 19}{2} = 190.$$

Step 2 — Favourable outcomes.

- Both balls black: C(10, 2) = (10·9)/2 = 45.
- Both balls white: C(10, 2) = 45.
- Total favourable = 45 + 45 = **90**.

Step 3 — Probability.

$$P(\text{same colour}) = \frac{90}{190} = \boxed{\tfrac{9}{19}}.$$

Key takeaway: "Same colour" splits naturally into two mutually exclusive sub-events. Add their probabilities.

Section C (any 3 of 5 × 14 = 42)

Q9 (14 marks) · Solve (D + 1)(D + D’ + 1) z = sin(x + 2y).

This is a linear PDE with constant coefficients. Solution = CF + PI.

Step 1 — Find the Complementary Function (CF).

The CF comes from `(D + 1)(D + D' + 1) z = 0`. Each linear factor contributes a piece.

For the factor (D + 1): write it as `D − 0·D' − (−1)`, so m = 0, c = −1. Its piece is

$$e^{cx}\phi(y+mx) = e^{-x}\,\phi_1(y).$$

For the factor (D + D’ + 1): write it as `D − (-1)D' − (−1)`, so m = −1, c = −1. Its piece is

$$e^{-x}\,\phi_2(y-x).$$

Therefore,

$$\text{CF} = e^{-x}\,\phi_1(y) + e^{-x}\,\phi_2(y-x),$$

with φ₁, φ₂ arbitrary functions.

Step 2 — Find the Particular Integral (PI).

We need z_p satisfying `(D + 1)(D + D' + 1) z_p = sin(x + 2y)`.

Since the operator F(D, D’) = (D + 1)(D + D’ + 1) contains odd-degree terms in D and D’, the safest method is to **try z_p = A sin(x + 2y) + B cos(x + 2y)** and solve for A, B by substitution.

First compute `(D + D' + 1) z_p`:

- D·sin(x+2y) = cos(x+2y); D·cos(x+2y) = −sin(x+2y).
- D’·sin(x+2y) = 2 cos(x+2y); D’·cos(x+2y) = −2 sin(x+2y).

So

$$(D+D'+1)[A\,\mathsf{sin}+B\,\mathsf{cos}] = A\,\mathsf{cos}+2A\,\mathsf{cos}+A\,\mathsf{sin}-B\,\mathsf{sin}-2B\,\mathsf{sin}+B\,\mathsf{cos} = (3A+B)\,\mathsf{cos}+(A-$$

Now apply (D + 1):

$$\begin{aligned}(D+1)\big[(3A+B)\,\mathsf{cos}(x+2y)+(A-3B)\,\mathsf{sin}(x+2y)\big] &= -\,(3A+B)\,\mathsf{sin}+(A-3B)\,\mathsf{cos}\\ &\quad+(3A+B)\,\mathsf{cos}+(A-3B)\,\mathsf{sin}\\ &=(-2A-4B)\,\mathsf{sin}(x+2y)+(4A-2B\end{aligned}$$

Setting this equal to sin(x + 2y):

$$-2A-4B=1, \qquad 4A-2B=0 \implies B=2A.$$

Substituting B = 2A in the first equation:

$$-2A - 8A = 1 \implies -10A = 1 \implies A = -\frac{1}{10}, \quad B = -\frac{1}{5}.$$

Hence

$$\text{PI} = -\frac{1}{10}\sin(x+2y) - \frac{1}{5}\cos(x+2y) = -\frac{1}{10}\Big[\sin(x+2y) + 2\cos(x+2y)\Big].$$

Step 3 — Complete solution.

$$\boxed{z = e^{-x}\,\phi_1(y) + e^{-x}\,\phi_2(y-x) - \frac{1}{10}\Big[\sin(x+2y) + 2\cos(x+2y)\Big]}$$

> **Key takeaway:** For sin/cos right-hand sides with operators that have odd-degree D or D' terms, the "guess A sin + B cos and substitute" method is more reliable than blindly substituting $D^2 \to -a^2$.

Q10 (14 marks) · Solve by Charpit's method: px + qy = pq.

Let

$$F(x,y,z,p,q) = px + qy - pq = 0.$$

Step 1 — Partial derivatives.

$$F_x = p, \quad F_y = q, \quad F_z = 0, \quad F_p = x - q, \quad F_q = y - p.$$

Step 2 — Charpit's auxiliary equations.

$$\frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q) + q(y-p)} = \frac{-dp}{p+0} = \frac{-dq}{q+0}.$$

Step 3 — Pick a clean pair. The last two ratios give

$$\frac{-dp}{p} = \frac{-dq}{q} \implies \frac{dp}{p} = \frac{dq}{q}.$$

Integrating: ln p = ln q + ln a (constant), so **p = a q**.

Step 4 — Use F = 0 with p = aq. Substitute into px + qy = pq:

$$aqx + qy = (aq)(q) = aq^2 \implies q(ax+y) = aq^2 \implies q = \frac{ax+y}{a}.$$

Then p = a q = a · (ax + y)/a = **ax + y**.

Step 5 — Integrate dz = p dx + q dy.

$$dz = (ax + y)\,dx + \frac{ax + y}{a}\,dy.$$

Let u = ax + y. Then du = a dx + dy, so dy = du − a dx. Substitute:

$$dz = u\,dx + \frac{u}{a}(du - a\,dx) = u\,dx + \frac{u}{a}\,du - u\,dx = \frac{u}{a}\,du.$$

Integrate:

$$z = \frac{u^2}{2a} + b = \frac{(ax + y)^2}{2a} + b.$$

Complete integral:

$$\boxed{z = \frac{(ax + y)^2}{2a} + b}$$

> **Key takeaway:** The cleanest pair of Charpit ratios in this problem was −dp/p = −dq/q, giving p = aq directly. Always try the dp, dq pair first.

Q11 (14 marks) · Find the mode from

Age	0–6	6–12	12–18	18–24	24–30
Frequency	7	10	24	**36**	23

Step 1 — Identify the modal class. The class with the maximum frequency is **18–24** (f_1 = 36).

Step 2 — Read off the values.

Symbol	Value
L (lower limit of modal class)	18
f_1 (modal class frequency)	36
f_0 (frequency of class before)	24
f_2 (frequency of class after)	23
h (class width)	6

Step 3 — Apply the formula.

$$\text{Mode} = L + h \cdot \frac{f_1 - f_0}{2f_1 - f_0 - f_2} = 18 + 6 \cdot \frac{36 - 24}{2(36) - 24 - 23} = 18 + 6 \cdot \frac{12}{72 - 47} = 18 + \frac{72}{25}.$$

$$\boxed{\text{Mode} = 18 + 2.88 = 20.88}$$

> **Key takeaway:** Find the modal class by inspection, then mechanically plug into the formula. Don't forget to multiply by h.

Q12 (14 marks) · The probability that a bomb dropped from a plane will strike the target is 1/3. If 6 bombs are dropped, find the probability that (i) exactly 2 will strike the target, (ii) at least 2 will strike the target.

Let X = number of bombs that strike the target. X ~ B(n, p) with n = 6, p = 1/3, q = 2/3.

(i) P(exactly 2 strike) — P(X = 2)

$$P(X=2) = \binom{6}{2} p^2\, q^4 = 15 \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^4 = 15 \cdot \frac{1}{9} \cdot \frac{16}{81}.$$

$$= \frac{15 \cdot 16}{9 \cdot 81} = \frac{240}{729} = \boxed{\frac{80}{243} \approx 0.329}.$$

(ii) P(at least 2 strike) — P(X ≥ 2)

Use the complement:

$$P(X \geq 2) = 1 - P(X=0) - P(X=1).$$

Compute P(X = 0) and P(X = 1):

$$P(X=0) = \binom{6}{0} p^0\, q^6 = \left(\frac{2}{3}\right)^6 = \frac{64}{729}.$$

$$P(X=1) = \binom{6}{1} p\, q^5 = 6 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^5 = 6 \cdot \frac{1}{3} \cdot \frac{32}{243} = \frac{192}{729} = \frac{64}{243}.$$

Therefore,

$$P(X \geq 2) = 1 - \frac{64}{729} - \frac{192}{729} = 1 - \frac{256}{729} = \boxed{\frac{473}{729} \approx 0.649}.$$

> **Key takeaway:** For "at least k" with small k, always use the complement — it's far less arithmetic than summing from k to n.

Q13 (14 marks) · Calculate the coefficient of correlation and obtain the lines of regression for the data

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

Step 1 — Compute means. n = 9. Σx = 1+2+…+9 = 45 ⇒ x̄ = 5. Σy = 9+8+10+12+11+13+14+16+15 = 108 ⇒ ȳ = 12.

Step 2 — Shift by means. Let X = x − 5, Y = y − 12.

x	y	X = x − 5	Y = y − 12	XY	X^2	Y^2
1	9	−4	−3	12	16	9
2	8	−3	−4	12	9	16
3	10	−2	−2	4	4	4
4	12	−1	0	0	1	0
5	11	0	−1	0	0	1
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	12	9	16
9	15	4	3	12	16	9
Σ		**0**	**0**	**57**	**60**	**60**

Step 3 — Karl Pearson's r.

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}} = \frac{57}{\sqrt{60 \cdot 60}} = \frac{57}{60} = \boxed{0.95}.$$

A very strong positive correlation.

Step 4 — Regression coefficients.

$$b_{yx} = \frac{\sum XY}{\sum X^2} = \frac{57}{60} = 0.95.$$

$$b_{xy} = \frac{\sum XY}{\sum Y^2} = \frac{57}{60} = 0.95.$$

Check: $r^2 = b\_yx \cdot b\_xy = 0.95 \times 0.95 = 0.9025 = (0.95)^2$. ✓

Step 5 — Regression lines.

Line of **y on x**:

$$y - \bar{y} = b_{yx}(x - \bar{x}) \implies y - 12 = 0.95(x - 5) \implies \boxed{y = 0.95\,x + 7.25}.$$

Line of **x on y**:

$$x - \bar{x} = b_{xy}(y - \bar{y}) \implies x - 5 = 0.95(y - 12) \implies \boxed{x = 0.95\,y - 6.4}.$$

Key takeaway: When means are integers, shift first — the columns become tidy and r drops out as a simple ratio. The two regression lines always intersect at ($\bar{x}$, $\bar{y}$) = (5, 12); check that as a sanity test.

Paper B — May 2024

Section A (5 × 2 = 10) — attempt all

Q1 (2 marks) · Form the partial differential equation from $z = f(x^2 + y^2)$.

Let $u = x^2 + y^2$, so $z = f(u)$. Differentiating partially:

$$p = \frac{\partial z}{\partial x} = 2x\, f'(u), \qquad q = \frac{\partial z}{\partial y} = 2y\, f'(u).$$

Dividing,

$$\frac{p}{q} = \frac{x}{y} \implies \boxed{py - qx = 0}.$$

Q2 (2 marks) · Write the one-dimensional heat equation.

The one-dimensional heat (or diffusion) equation is

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

where $u(x, t)$ is the temperature at position x and time t in a uniform rod, and c^2 is the thermal diffusivity of the material.

Q3 (2 marks) · Write the wave equation.

The one-dimensional wave equation is

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

where $y(x, t)$ is the transverse displacement of a vibrating string at position x and time t, and $c^2 = T/\rho$ (tension over linear mass density).

Q4 (2 marks) · Write Lagrange's Linear Equation.

Lagrange's linear partial differential equation is the first-order linear PDE of the form

$$\boxed{P\,p + Q\,q = R}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, and P, Q, R are functions of x, y, z. Its general solution is found from the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

giving two independent integrals $u = c_1$, $v = c_2$, with general solution $\varphi(u, v) = 0$.

Q5 (2 marks) · Find the probability of getting 2 heads in 3 tosses of a fair coin.

Tossing a fair coin 3 times has $2^3 = 8$ equally likely outcomes. The outcomes with exactly 2 heads are HHT, HTH, THH — that's 3.

$$P(\text{2 heads}) = \frac{3}{8}.$$

Alternatively, by the Binomial formula with $n = 3$, $p = 1/2$:

$$P(X = 2) = \binom{3}{2} \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^1 = 3 \cdot \tfrac{1}{8} = \boxed{\tfrac{3}{8}}.$$

Section B (any 2 of 3 × 9 = 18)

Q6 (9 marks) · Solve $p^2 + q^2 = 1$.

This is Standard Form I — a non-linear first-order PDE involving only p and q.

Step 1 — Try $z = ax + by + c$. Then $p = a$, $q = b$.

Step 2 — Substitute. The PDE $p^2 + q^2 = 1$ becomes

$$a^2 + b^2 = 1 \implies b = \sqrt{1 - a^2}.$$

Step 3 — Complete integral.

$$\boxed{z = a\,x + \sqrt{1 - a^2}\;\,y + c}$$

with a (chosen freely from $|a| \leq 1$) and c being the two arbitrary constants.

Singular integral: Eliminate a and c from z, ∂z/∂a, ∂z/∂c. Since ∂z/∂c = 1 ≠ 0, no singular integral exists.

> **Key takeaway:** Recognise Form I (only p, q in the PDE), substitute the plane $z = ax + by + c$, and reduce the PDE to an algebraic relation between a and b.

Q7 (9 marks) · Find the first moment about the origin for Binomial Distribution.

Let X ~ B(n, p) with q = 1 − p. The probability mass function is

$$P(X = k) = \binom{n}{k} p^k\, q^{n-k}, \qquad k = 0, 1, \ldots, n.$$

The **r-th moment about the origin** is defined as μ_r′ = E[X^r]. So the **first moment about the origin** is

$$\mu_1' = E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k\, q^{n-k}.$$

Derivation: Use the identity `k · C(n, k) = n · C(n − 1, k − 1)` (valid for k ≥ 1; the k = 0 term is zero anyway):

$$\mu_1' = \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k q^{n-k} = np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k}.$$

Let j = k − 1:

$$\mu_1' = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np\,(p+q)^{n-1}.$$

Since p + q = 1:

$$\boxed{\mu_1' = np}.$$

Interpretation: The first moment about the origin equals the mean. For Binomial, mean = np. That's the expected number of successes in n trials.

> **Key takeaway:** The key identity `k · C(n, k) = n · C(n − 1, k − 1)` is the standard trick to compute Binomial moments.

Q8 (9 marks) · Find the mean of the distribution

Class	0–10	10–20	20–30	30–40
Frequency	8	7	20	5

Step 1 — Build the table.

Class	Midpoint x	Frequency f	f·x
0–10	5	8	40
10–20	15	7	105
20–30	25	20	500
30–40	35	5	175
Total		**Σf = 40**	**Σfx = 820**

Step 2 — Apply the mean formula.

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{820}{40} = \boxed{20.5}.$$

> **Key takeaway:** Same procedure as Paper A Q7. Always build the table — it both organises your work and earns step marks even if the arithmetic slips.

Section C (any 3 of 5 × 14 = 42)

Q9 (14 marks) · Solve (D + 1)(D + D' − 1) z = sin(x + 2y).

Solution = CF + PI.

Step 1 — Complementary Function.

For the factor (D + 1): writing it as `D − 0·D' − (−1)`, we get m = 0, c = −1. Its CF piece is `e^{−x} φ₁(y)`.

For the factor (D + D' − 1): writing it as `D − (−1)D' − 1`, we get m = −1, c = 1. Its CF piece is `e^{x} φ₂(y − x)`.

Therefore,

$$\text{CF} = e^{-x}\,\phi_1(y) + e^{x}\,\phi_2(y-x).$$

Step 2 — Particular Integral.

Because F(D, D') = (D+1)(D+D'−1) contains odd-degree terms in D and D', use the "guess form, substitute" method. Assume

$$z_p = A\,\text{sin}(x+2y) + B\,\text{cos}(x+2y).$$

Apply (D + D' − 1) first:

- D[A sin + B cos] = A cos − B sin
- D'[A sin + B cos] = 2A cos − 2B sin
- −1·[A sin + B cos] = −A sin − B cos

Sum:

$$(D+D'-1)[A\,\text{sin}+B\,\text{cos}] = (3A-B)\,\text{cos}(x+2y)+(-A-3B)\,\text{sin}(x+2y)-\text{wait, let me redo.}$$

Let me be careful:

- Coefficient of cos: A + 2A = 3A, plus from −1·(B cos) we get −B. So total cos coefficient = 3A − B.
- Coefficient of sin: −B − 2B = −3B, plus from −1·(A sin) we get −A. So total sin coefficient = −A − 3B.

So `(D + D' − 1)[A sin + B cos] = (−A − 3B) sin(x+2y) + (3A − B) cos(x+2y)`.

Now apply (D + 1):

- D[(−A − 3B) sin + (3A − B) cos] = (−A − 3B) cos − (3A − B) sin
- 1·[(−A − 3B) sin + (3A − B) cos] = (−A − 3B) sin + (3A − B) cos

Sum: - Coefficient of sin: −(3A − B) + (−A − 3B) = −3A + B − A − 3B = −4A − 2B. - Coefficient of cos: (−A − 3B) + (3A − B) = 2A − 4B.

So `(D + 1)(D + D' − 1)[A sin + B cos] = (−4A − 2B) sin(x+2y) + (2A − 4B) cos(x+2y)`.

Setting this equal to sin(x + 2y):

$$-4A - 2B = 1, \qquad 2A - 4B = 0 \implies A = 2B.$$

Substituting A = 2B into the first equation:

$$-4(2B) - 2B = 1 \implies -10B = 1 \implies B = -\tfrac{1}{10}, \quad A = -\tfrac{1}{5}.$$

Therefore,

$$\text{PI} = -\tfrac{1}{5}\sin(x+2y) - \tfrac{1}{10}\cos(x+2y) = -\tfrac{1}{10}\Big[2\sin(x+2y) + \cos(x+2y)\Big].$$

Step 3 — Complete solution.

$$\boxed{z = e^{-x}\,\phi_1(y) + e^{x}\,\phi_2(y-x) - \tfrac{1}{10}\big[\,2\sin(x+2y) + \cos(x+2y)\,\big]}$$

> **Key takeaway:** Note how this question differs from Paper A’s version by a single sign — that flips one of the CF exponentials from e^{−x} to e^{+x}. Read carefully!

Q10 (14 marks) · Solve by Charpit’s method: (p² + q²)y = qz.

Let F(x, y, z, p, q) = (p² + q²)y − qz = 0.

Step 1 — Partial derivatives.

$$F_x = 0, \quad F_y = p^2 + q^2, \quad F_z = -q, \quad F_p = 2py, \quad F_q = 2qy - z.$$

Step 2 — Charpit’s auxiliary equations.

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + q(2qy - z)} = \frac{-dp}{0 + p(-q)} = \frac{-dq}{(p^2+q^2) + q(-q)}.$$

Simplify the last two:

$$\frac{-dp}{-pq} = \frac{-dq}{p^2}, \quad \text{i.e.,} \quad \frac{dp}{pq} = \frac{-dq}{p^2}.$$

Step 3 — Cross-multiply.

$$p^2\,dp = -pq\,dq \implies p\,dp + q\,dq = 0 \implies \tfrac{1}{2}d(p^2+q^2) = 0.$$

Integrating, **p² + q² = a²** (a constant).

Step 4 — Substitute back into F = 0.

$$(p^2+q^2)\,y = q\,z \implies a^2\,y = q\,z \implies q = \frac{a^2 y}{z}.$$

Then

$$p^2 = a^2 - q^2 = a^2 - \frac{a^4y^2}{z^2} = \frac{a^2(z^2-a^2y^2)}{z^2},$$

so

$$p = \frac{a}{z}\sqrt{z^2-a^2y^2}.$$

Step 5 — Integrate dz = p dx + q dy.

$$dz = \frac{a\sqrt{z^2-a^2y^2}}{z}\,dx + \frac{a^2y}{z}\,dy.$$

Multiply through by z:

$$z\,dz = a\sqrt{z^2-a^2y^2}\,dx + a^2y\,dy \implies z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx.$$

Notice that

$$z\,dz - a^2y\,dy = \tfrac{1}{2}\,d(z^2-a^2y^2).$$

Let $u = z^2 - a^2y^2$. Then $(1/2)\,du = a\sqrt{u}\,dx$, i.e.,

$$\frac{du}{2\sqrt{u}} = a\,dx \implies \sqrt{u} = ax + b.$$

Squaring:

$$\boxed{z^2 - a^2y^2 = (ax+b)^2 \iff z^2 = (ax+b)^2 + a^2y^2}$$

with a and b the two arbitrary constants.

> **Key takeaway:** The clean ratio in Charpit was `dp/(pq) = −dq/p²`, which integrated immediately to `d(p² + q²) = 0`. The substitution `u = z² − a²y²` then handles the integration mechanically.

Q11 (14 marks) · Find the mode from

Age	0–6	6–12	12–18	18–24	24–30
Frequency	6	11	25	**35**	13

Step 1 — Modal class. Maximum frequency = 35, so the modal class is **18–24**.

Step 2 — Values. $L = 18$, $f_1 = 35$, $f_0 = 25$, $f_2 = 13$, $h = 6$.

Step 3 — Apply the formula.

$$\text{Mode} = L + h \cdot \frac{f_1 - f_0}{2f_1 - f_0 - f_2} = 18 + 6 \cdot \frac{35 - 25}{2(35) - 25 - 13} = 18 + 6 \cdot \frac{10}{32} = 18 + \frac{60}{32}.$$

$$\boxed{\text{Mode} = 18 + 1.875 = 19.875}$$

Key takeaway: Identical procedure to Paper A Q11 — only the numbers change. This is *the* repeating template.

Q12 (14 marks) · Calculate the correlation coefficient between x and y for

x	21	30	54	57	72	78	90
y	60	72	83	110	100	92	135

The means are not integers, so we use the **direct (no-shift) formula**:

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left[n\sum x^2 - (\sum x)^2\right]\left[n\sum y^2 - (\sum y)^2\right]}}, \qquad n = 7.$$

Step 1 — Build the table.

x	y	x^2	y^2	$x \cdot y$
21	60	441	3600	1260
30	72	900	5184	2160
54	83	2916	6889	4482
57	110	3249	12100	6270
72	100	5184	10000	7200
78	92	6084	8464	7176
90	135	8100	18225	12150
Σ				
$\Sigma x = 402$	$\Sigma y = 652$	$\Sigma x^2 = 26874$	$\Sigma y^2 = 64462$	$\Sigma xy = 40698$

Step 2 — Compute the numerator.

$$n\sum xy - \sum x \cdot \sum y = 7(40698) - (402)(652) = 284886 - 262104 = 22782.$$

Step 3 — Compute the two denominator factors.

$$n\sum x^2 - (\sum x)^2 = 7(26874) - (402)^2 = 188118 - 161604 = 26514.$$

$$n\sum y^2 - (\sum y)^2 = 7(64462) - (652)^2 = 451234 - 425104 = 26130.$$

Step 4 — Combine.

$$r = \frac{22782}{\sqrt{26514 \cdot 26130}} = \frac{22782}{\sqrt{692{,}810{,}820}} = \frac{22782}{26321.3} \approx \boxed{0.866}.$$

A strong positive correlation: as x grows, y tends to grow too.

> **Key takeaway:** When means aren't integers, don't try to shift — the direct formula is fastest. Build the five-column table (x, y, x², y², xy), then plug.

Q13 (14 marks) · The mean and variance of a binomial distribution are 4 and 2 respectively. Find P(X = 2), P(X < 2), P(X ≥ 2).

Step 1 — Determine n and p. Given mean = np = 4 and variance = npq = 2.

$$q = \frac{\text{variance}}{\text{mean}} = \frac{2}{4} = \tfrac{1}{2} \implies p = 1 - q = \tfrac{1}{2}.$$

$$n = \frac{\text{mean}}{p} = \frac{4}{1/2} = 8.$$

So X ~ B(8, 1/2).

Step 2 — (i) Exactly two successes: P(X = 2).

$$P(X = 2) = \binom{8}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^6 = 28 \cdot \left(\tfrac{1}{2}\right)^8 = \frac{28}{256} = \boxed{\tfrac{7}{64} \approx 0.109}.$$

Step 3 — (ii) Less than two successes: P(X < 2) = P(X = 0) + P(X = 1).

$$P(X = 0) = \binom{8}{0}\left(\tfrac{1}{2}\right)^8 = \tfrac{1}{256}, \qquad P(X = 1) = \binom{8}{1}\left(\tfrac{1}{2}\right)^8 = \tfrac{8}{256}.$$

$$P(X < 2) = \tfrac{1}{256} + \tfrac{8}{256} = \boxed{\tfrac{9}{256} \approx 0.035}.$$

Step 4 — (iii) At least two successes: P(X ≥ 2).

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \tfrac{9}{256} = \boxed{\tfrac{247}{256} \approx 0.965}.$$

Key takeaway: From mean and variance to n and p is a two-line algebraic step. After that, the three probabilities follow from one PMF formula and the complement rule.

Part 5 — Back Matter

Quick Reference Card (one-spread cheat sheet)

Partial Differential Equations

Type	Form	Method
Linear (Lagrange)	$Pp + Qq = R$	Aux. eqns $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
Standard Form I	$f(p, q) = 0$	Try $z = ax + by + c$, with $f(a, b) = 0$
Standard Form II	$z = px + qy + f(p, q)$ (Clairaut)	$z = ax + by + f(a, b)$
Standard Form III	$f_1(x, p) = f_2(y, q)$	Set each side $= a$, solve, integrate
Standard Form IV	General nonlinear	**Charpit's method**
Linear const. coeff.	$F(D, D')z = \phi(x, y)$	CF (from auxiliary eqn) + PI (operator method)

Forming a PDE from $z = f(x^2 + y^2)$

$$\boxed{py - qx = 0}$$

Path: $p = 2xf'$, $q = 2yf'$ $\Rightarrow$ eliminate f' by ratio.

Classical PDEs

Equation	Form	Physical meaning
1D heat	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	Temperature in a rod
2D heat	$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$	Temperature in a plate
1D wave	$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$	Vibrating string
Laplace	$\nabla^2 u = 0$	Steady-state temperature / potential

Fourier Transforms

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin(sx)\, dx \qquad F_c\{f(x)\} = \int_0^\infty f(x) \cos(sx)\, dx$$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Statistics — Mean / Median / Mode

Measure	Formula
Mean (grouped)	$\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$
Median (grouped)	$L + \dfrac{N/2 - F}{f} \cdot h$
Mode (grouped)	$L + \dfrac{f_1 - f_0}{2f_1 - f_0 - f_2} \cdot h$

Correlation and Regression

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \cdot \sum(y_i - \bar{y})^2}}$$

$$b_{yx} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}, \quad b_{xy} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2}$$

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

Regression lines:

- y on x: $y - \bar{y} = b_{yx}(x - \bar{x})$
- x on y: $x - \bar{x} = b_{xy}(y - \bar{y})$

Binomial Distribution

$$P(X = r) = \binom{n}{r} p^r q^{n-r}, \quad q = 1 - p$$

- Mean $= np$
- Variance $= npq$
- SD $= \sqrt{npq}$
- First moment about origin $= np$

Top Mistakes to Avoid in the Exam

These are subject-specific traps. Read this list once an hour before you walk into the hall.

PDE Section

1. **Confusing p and q.** $p = \partial z/\partial x$, $q = \partial z/\partial y$. Mixing them up costs the whole question. Write "p is $\partial z/\partial x$" in the margin before starting.
2. **In Charpit, forgetting that a is the arbitrary constant from $F = a$**, not a random letter. The whole point is that integrating one auxiliary ratio gives you a relation between p, q and an arbitrary constant a.
3. **PI for** sin$(\alpha x + \beta y)$ **with operator D^2 or DD':** replace $D^2 \to -\alpha^2$, $DD' \to -\alpha\beta$, ${D'}^2 \to -\beta^2$. Sign errors here are very common.
4. **When the operator gives 0 after substitution** (i.e. the function is a solution of the homogeneous equation), the standard PI formula fails. You need to multiply by x and differentiate the operator. Watch for this.
5. **Writing the PDE without eliminating the arbitrary function/constant.** "Forming the PDE from $z = f(\ldots)$" means the answer must NOT contain f or f'. If f is still there, you haven't finished.

Statistics Section

6. **Mode formula sign:** denominator is $2f_1 - f_0 - f_2$, NOT $f_0 - f_2$. The "$2f_1$" piece is the key, and the signs in front of f_0 and f_2 are both minus.
7. **Identifying the modal class wrong.** The modal class is the one with the **highest frequency**, period. Don't pick the middle class because it "looks" balanced.
8. **For grouped mean, x_i** is the **midpoint** of each class, not the upper or lower bound. So for class 0–10, $x_i = 5$.
9. **In correlation, mixing up b_{yx} and b_{xy}:**
   - b_{yx} has $\sum(x - \bar{x})^2$ in the denominator (slope of y on x)
   - b_{xy} has $\sum(y - \bar{y})^2$ in the denominator (slope of x on y)
10. **Sign of r when computing $\sqrt{b_{yx} b_{xy}}$.** The sign of r matches the sign of either regression coefficient (they're always the same sign as r). If both b's are positive, r is positive.

Probability Section

11. **Binomial: "at least 2" means $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$.** Don't compute $P(X = 2) + P(X = 3) + \ldots$ — way too much work.
12. **"Mean and variance given, find n and p":** use $np =$ mean and $npq =$ var. Divide to get q, then $p = 1 - q$, then $n =$ mean$/p$.
13. **"Same colour" doesn't mean "both black" alone.** It means both black OR both white — sum the two probabilities.

14. **Probability of an odd number on a die = 3/6 = 1/2**, not 1/3 or 1/6. There are three odd faces: 1, 3, 5.

General Exam Discipline

15. **Always state the formula before plugging in numbers.** Examiners give partial marks for formulas even if your arithmetic goes wrong.
16. **Box your final answer.** Use $\boxed{\ldots}$ or a clear underline. Makes the examiner's job easier and your answer more visible.
17. **Don't skip Section A.** It's only 10 marks but it's the easiest 10 marks in the paper. Two-mark questions need a one-line answer with the formula, not a derivation.
18. **In Section C, attempt only 3 questions.** Attempting 4 won't get you extra marks but will eat your time. Pick the 3 you're strongest at.

One-Page Last-Minute Cheat Sheet

Glance at this 15 minutes before the exam. Don't try to learn anything new.

PDE essentials

- $p = z_x,\ q = z_y$
- Forming PDE: eliminate f / arbitrary constants by differentiating w.r.t. x and y
- $z = f(x^2 + y^2) \Rightarrow py - qx = 0$
- Lagrange: $Pp + Qq = R \Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
- $f(p, q) = 0$: complete integral is $z = ax + by + c$ with $f(a, b) = 0$
- Charpit auxiliary: $\frac{dp}{F_x+pF_z} = \frac{dq}{F_y+qF_z} = \frac{dz}{-pF_p-qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$

Operator method PI

- For e^{ax+by}: replace $D \to a$, $D' \to b$
- For sin$(\alpha x + \beta y)$ or cos: replace $D^2 \to -\alpha^2$, $DD' \to -\alpha\beta$, ${D'}^2 \to -\beta^2$
- If denominator becomes 0: multiply numerator by x, differentiate denominator w.r.t. D

Standard PDEs

- 1D heat: $u_t = c^2 u_{xx}$
- 2D heat: $u_t = c^2(u_{xx} + u_{yy})$
- 1D wave: $y_{tt} = c^2 y_{xx}$

Stats

- Mean (grouped): $\bar{x} = \sum f_i x_i / \sum f_i$, x_i = midpoint
- Mode: $L + \frac{f_1-f_0}{2f_1-f_0-f_2} h$ (L = lower bound of modal class, h = class width)
- Correlation: $r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}}$
- Regression y on x: $y - \bar{y} = b_{yx}(x - \bar{x})$, $b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$

Binomial

- $P(X = r) = \binom{n}{r} p^r q^{n-r}$
- Mean np, variance npq
- "At least 2": $1 - P(0) - P(1)$
- Given mean μ and variance σ^2: $q = \sigma^2/\mu$, $p = 1 - q$, $n = \mu/p$

Probability basics

- P(odd on die) $= 1/2$
- P(2 heads in 3 tosses) $= \binom{3}{2}(1/2)^3 = 3/8$
- Urn (2 same colour) $= P($both B$) + P($both W$)$

Fourier

- Sine transform: $F_s\{f\} = \int_0^\infty f(x)\,\sin(sx)\,dx$
- Cosine transform: $F_c\{f\} = \int_0^\infty f(x)\,\cos(sx)\,dx$

Closing Note

You've got everything you need.

The pattern from the last two papers is *boringly consistent*. Same Q1 every time (form PDE from $z = f(x^2 + y^2)$). Same Q9 family (linear PDE with operators, the sin$(x + 2y)$ variant on the right). Same Q11 family (mode from a 5-bucket frequency table). Same statistics rotation (mean, correlation, binomial).

If you can do the 13 PYQ solutions in this document cold — recognize the question type, write the formula, push through the arithmetic — you have an 80%+ paper locked in.

A few tactical reminders:

- **Walk in with the cheat sheet's formulas memorized.** The 2-mark questions are pure recall.
- **Start with Section A.** Get the easy 10 marks in 15 minutes, then move to whichever Section C question you're most confident on.
- **In Section C, write the formula first, the substitution second, the calculation third.** Even if your arithmetic fails, you're getting partial marks for structure.
- **Don't second-guess the regression / correlation question — set up the table, compute the means, fill in the deviations.** It's mechanical once the table is up.

Three days from now, you'll be done with this paper. Sleep well the night before, eat something solid in the morning, and trust the reps.

You've put in the work. Go finish it.

— Math IV prep, written for you